CALCULATIONS FOR CAVITATING FLOWS

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A numerical method is proposed for solving the following problems in the hydrodynamics of an ideal incompressible fluid with free boundaries: the problem of constructing a profile from the pressure prescribed on it; the problem of constructing a cavern behind an arbitrary arc and then closing the cavern by an artificial curve whose shape is not a priori known; and Ryabushinskii's [1] problem in the plane and axisymmetric cases. All two-dimensional problems for the upper half-plane are solved with allowance for gravitational force; Ryabushinskii's problem for the axisymmetric case is solved with allowance for surface-tension forces. A solution to Ryabushinskii's two-dimensional problem with allowance for surface-tension and gravitational forces (for the upper half-plane) has been obtained in [2].

The algorithm for solving the aforesaid two-dimensional problems is similar in many ways to the one proposed in [2]. It differs only in the method used for correcting the free boundary, which is of prime importance in the solution of problems of this type. Earlier publications dealing with the numerical solution of problems in hydrodynamics with free boundaries (see, for example, [3, 4]), as a rule, lack a description of free-boundary correction techniques.

The advantage of numerical methods of solving two-dimensional problems with free boundaries over analytical methods based on the apparatus of the theory of analytic functions is that numerical methods can be applied without appreciable changes to analogous axisymmetric and strictly three-dimensional problems. As an example, the calculation of Ryabushinskii's axisymmetric problem is given in the final section of this paper. As distinct from Garabedian's [5] approximate method, the method proposed makes it possible to obtain a solution to this problem for a relatively wide class of "cavitator" configurations.

1. The method of solution is illustrated in detail by its application to the problem of constructing a plane profile from the pressure prescribed on it. This problem, without allowance for gravitational forces, has been solved by Mal'tsev [6]. In the following, we assume everywhere a potential steady flow. In dimensionless form, the equations for such a flow are

$$\frac{\partial\psi}{\partial x} = -\frac{\partial\varphi}{\partial z} = -V_z, \quad \frac{\partial\psi}{\partial z} = \frac{\partial\varphi}{\partial x} = V_x, \tag{1.1}$$

$$P = \frac{2(p-p_0)}{\rho V_0^2} = 1 - V^2 - Gz, \quad G = \frac{2gS}{V_0^2}, \tag{1.2}$$

where V_0 and p_0 are the velocity and pressure of the oncoming flow along the x-axis; V and p are the velocity and pressure in the flow; g is the acceleration of the gravitational force directed along the z-axis; and S is a certain linear dimension. The cavitation number σ is related to the pressure P by the relationship $\sigma = -P^\circ$, where P° is the pressure in the cavern.

The solution of the problem is sought within the rectangle D formed (Fig. 1) by the x-axis, the required profile, the straight lines C_1, C_3 (parallel to the z-axis at a distance γ, to either side of the origin of the coordinates), and the streamline C_2 that passes through the point (γ, R). The values of the parameters γ and R are obtained from the asymptotic form of the flow: the required profile is replaced by a circle of the same diameter, and the parameters γ and R are selected, such that the velocity component V_x at the points $(\gamma, 0)$ and $(0, R)$ differs from the velocity of the oncoming flow by a small prescribed value. Neglecting this value, we have

$$V_x = 1 \text{ на } C_1, C_2, C_3. \tag{1.3}$$

If the x-axis and the required profile are taken as the zero streamline, then with the aid of the second equation in (1.1), condition (1.3) can be easily reduced to the form

$$\psi = z \text{ on } C_1, C_3, \ \psi = R \text{ on } C_2, \quad \psi = 0 \text{ on } C. \tag{1.4}$$

It is convenient to solve the problem in the auxiliary plane $x\psi$, since in this plane the range of variation of the variables is known. The function $z(x, \psi)$ will be considered as the one to be determined. By changing to the variables x, ψ, we get for z the following equation:

$$az_{xx} - 2bz_{x\psi} + cz_{\psi\psi} = 0,$$
$$a = z_\psi^2, \qquad b = z_x z_\psi, \qquad c = 1 + z_x^2. \tag{1.5}$$

Expressed in terms of the function $z(x, \psi)$, the formula for the flow rate is

$$V^2 = \frac{1 + z_x^2}{z_\psi^2}. \tag{1.6}$$

In this case, the region D becomes a rectangle $\omega(-\gamma \le x \le \gamma, \ 0 \le \psi \le R)$. It should be noted that a singularity of the Jacobian of the transformation at the stagnation points of the flow affects the solution of the problem only in an insignificant region about these points.

Thus, the problem reduces to the solution of Eq. (1.5) in the region ω for the following boundary conditions (Fig. 2)

$$z = \psi \text{ on } \Gamma \tag{1.7}$$

$$P(x) - 1 + \frac{1 + z_x^2}{z_\psi^2} + Gz = 0 \quad \text{on } \Gamma^* \tag{1.8}$$

where P(x) is a given pressure.

The algorithm for solving the problem consists in alternately integrating Eq. (1.5) for a fixed free boundary and the boundary conditions (1.7), and in converting the free boundary according to condition (1.8). Iteration is discontinued as soon as the required values and the flow characteristics cease to vary. Their limiting value is taken as the solution of the problem [2].

Integration of Eq. (1.5) is performed with the aid of fractional step schemes which provide complete approximation; an example is a scheme proposed by Douglas [7, 8]:

$$\tau^{-1}(z^{n+1/2} - z^n) = \varepsilon\,[a^n\Lambda_{11}z^{n+1/2} - (2b^n\Lambda_{12} - c^n\Lambda_{22})\,z^n] +$$
$$+ (1-\varepsilon)\,[(a^n\Lambda_{11} - 2b^n\Lambda_{12})\,z^n + c^n\Lambda_{22}z^{n+1/2}],$$
$$\tau^{-1}(z^{n+1} - z^{n+1/2}) = [\varepsilon a^n\Lambda_{22} + (1-\varepsilon)a^n\Lambda_{11}](z^{n+1} - z^n), \tag{1.9}$$

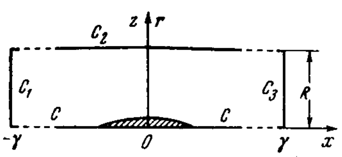

Fig. 1

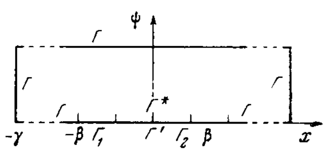

Fig. 2

or the stabilizing operator scheme [7, 9, 10]

$$\tau^{-1}(E-\mu\tau\Lambda_{11})(E-\mu\tau\Lambda_{22})(z^{n+1}-z^n)= \\ =(a^n\Lambda_{11}-2\,b^n\Lambda_{12}+c^n\Lambda_{22})\,z^n\,. \tag{1.10}$$

Here, Λ_{11}, Λ_{12}, Λ_{22} are central difference operators, and E is a unitary operator. The schemes are realized by three-point filtering in the x- and ψ-directions. A ruled surface is taken as the initial integral surface

$$z^\circ(x,\psi)=\left(1-\frac{z^\circ(x,0)}{R}\right)\psi+z^\circ(x,0), \tag{1.11}$$

where $z^\circ(x,0)$ is the equation for the initial zero streamline.

The free surface is converted with the aid of the following finite-difference representation of the Bernoulli integral (1.8):

$$1+(z_x^k)^2+(P-1+Gz)^k z_\psi^k(\xi z^{k+1}+z_\psi^k-\xi z^k)=0, \tag{1.12}$$

$$P^k=P(x)\chi(k), \tag{1.13}$$

$$z_x=\frac{z_{i+10}-z_{i-10}}{h_i+h_{i+1}},\qquad z_\psi=\xi z_{i0}+\eta z_{i1}+\zeta z_{i2},$$

$$\xi=-(\eta+\zeta),\ \eta=(l_1+l_2)/l_1l_2,\ \zeta=-1/\eta\,l_2^2. \tag{1.14}$$

Here, h_i is the step along the x-axis; l_j is the step along the ψ-axis; k is the profile correction number; and $\chi(k)$ is a positive monotonically increasing function that satisfies the conditions

$$\chi(0)=0,\ \lim\chi(k)=1,\ \text{for } k\to\infty.$$

Instead of the explicit scheme (1.12), an implicit scheme [2]

$$1+z_x^k z_x^{k+1}+(P-1+Gz)^k z_\psi^k(\xi z^{k+1}+z_\psi^k-\zeta z^k)=0 \tag{1.15}$$

can be applied to the correction of the profile, in which correction is achieved by once-through filtering of this relation along the free boundary.

The computational procedure of the problem is as follows: setting $z^\circ(x,0)=0$, Eq. (1.5) is integrated on the basis of scheme (1.9) or (1.10). Then, the first correction of the profile is performed either implicitly by formula (1.12), or by filtering according to scheme (1.15). This is followed by successively repeating the following computational cycle: integration of Eq. (1.5) on the basis of scheme (1.9) or (1.10) with boundary conditions (1.7) and the condition

$$z=z^k(x,0)\ \text{on}\ \Gamma^*, \tag{1.16}$$

where $z^k(x,0)$ is the profile equation after the k-th correction and the correction of the profile according to condition (1.8) as realized on the basis of (1.12) or (1.15). Calculations are discontinued after "stabilization."

Integration of (1.5) (solution of the Dirichlet problem) need not be complete—two or three iterations with respect to the parameter τ are sufficient. Calculations were performed for following values of the problem parameters: $\tau=0.2$, $\gamma=10\beta$, $\beta=2.5$, $R=10$. The function $\chi(k)$ was selected in the form

$$\chi(k)=\frac{\nu^k-1}{\nu^k},\qquad \nu=1.04. \tag{1.17}$$

The grid in the plane (x,ψ) is nonuniform; the grid in the γ-direction becomes increasingly dense toward the x-axis, the grid in the x-direction becomes increasingly dense at the stagnation points of the flow and at the ends of a cavern in problems involving caverns.*

The method was verified by calculating the following problem. We took an analytical solution, obtained by applying the "mirror" scheme [11, 12] to the problem of the cavitating flow of a weightless fluid past a wedge. On its basis, the pressure between the stagnation points was

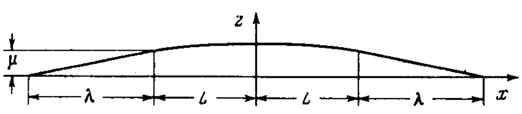

Fig. 3

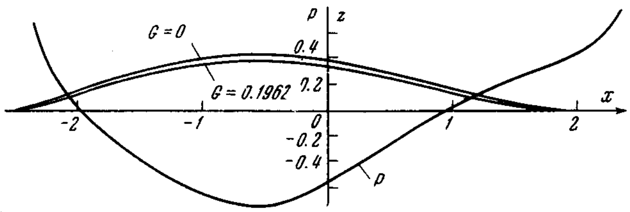

Fig. 4

calculated and was used to plot the profile. The geometry of the problem was as follows (see Fig. 3): $\lambda=1$, $\mu=0.2$, $\sigma=0.4$. The calculations (see Table 1) were performed twice, first with scheme (1.12) and then scheme (1.15) for correcting the profile. Discrepancies between the results were observed only in the fourth decimal place (the table gives the results obtained with the aid of scheme (1.12)). The last two columns in the table give the results obtained with allowance for a gravity force $G=0.1962$. For $S=1$ m, this corresponds to a velocity of 10 m/sec for the oncoming flow.

Figure 4 shows the results of profile computations from an arbitrarily given pressure (the pressure curve is plotted by hand on millimeter graph paper) with and without allowance for the gravitational force.

2. The problem of constructing a cavern of prescribed length behind an arbitrary arc can be solved in the same manner. The cavern is closed by means of an artificial curve whose shape is not given a priori. The problem is solved by the general scheme described.

Let $W(x)$, $x\in(-\beta,-\alpha)$ (Fig. 5) be the equation of the cavitator arc. We take the straight line between the points $(-\alpha,\mu)$ and $(\beta',0)$ as the initial profile.

The free boundary (boundary of the cavern and closing curve) will be corrected by either scheme (1.12) or (1.15), evaluating

$$Q^k=P^k-1.$$

from formula

$$Q^k=Q^{*k},\quad x\in(-\alpha,\alpha'),\quad Q^k=Q^{*k}\cdot\nu(x),\quad x\in(\alpha',\beta'), \tag{2.1}$$

$$Q^{*k}=-\left[Gz+\frac{1+z_x^2}{z_\psi(z_\psi+\xi(z^*-z))}\right]^k_{x=-\alpha+h},$$

$$z^*=W(-\alpha)+hW'(-\alpha). \tag{2.2}$$

where h is the step on the x-axis, adjacent from the right to the point $-\alpha$; z_x and z_ψ have the form (1.14); and ν_x is a positive monotone function with a continuous derivative, for which

$$\nu(\alpha')=1,\qquad \nu(\beta')=0.$$

Condition (2.2) is the condition for smooth contact between the free surface and the cavitator, while (2.1) is the condition for constant pressure in the cavern.

Figure 5 shows the shapes of the cavern and closing curve for two arcs (cavitators), an inclined straight line and an elliptic arc, respectively,

$$W(x)=\mu(\beta+x)/(\beta-\alpha),$$

$$W(x)=\mu\sqrt{(\beta^2-x^2)/(\beta^2-\alpha^2)}.$$

The following assumptions were made for the computations:

$$\nu(x)=(\beta'-x)/(\beta'-\alpha'),\ \alpha=\alpha'=1.5,$$

$$\beta=\beta'=2.5,\ \mu=0.5773.$$

The free surface was corrected according to scheme (1.15). The cavitation number σ is equal to 0.8970 and 0.6969, respectively. The gravitational forces was neglected in the calculations.

3. The problem of the cavitating flow of a heavy fluid near an arbitrary arc according to the mirror scheme (Ryabushinskii's problem) is a special case of the problem examined above. Figure 6 shows the

*In test computations (Table 1), the grid in the x-direction is plotted from an analytical solution.

Table 1

Analytical solution			Numerical solution			
x	z	V	z	V	z	V
0.000	0.2540	1.1832	0.2518	1.1832	0.2242	1.1644
0.026	0.2540	1.1832	0.2517	1.1832	0.2241	1.1644
0.080	0.2536	1.1832	0.2513	1.1832	0.2233	1.1645
0.159	0.2523	1.1832	0.2501	1.1832	0.2227	1.1645
0.261	0.2494	1.1832	0.2473	1.1832	0.2202	1.1648
0.380	0.2441	1.1832	0.2423	1.1832	0.2157	1.1651
0.506	0.2360	1.1832	0.2347	1.1832	0.2039	1.1657
0.625	0.2256	1.1832	0.2250	1.1832	0.2001	1.1665
0.724	0.2142	1.1832	0.2146	1.1832	0.1906	1.1673
0.792	0.2044	1.1832	0.2060	1.1832	0.1827	1.1679
0.817	0.2000	1.1832	0.2019	1.1832	0.1790	1.1682
1.095	0.1444	1.0475	0.1423	1.0475	0.1230	1.0359
1.303	0.1027	0.9986	0.0994	0.9986	0.0838	0.9904
1.465	0.0703	0.9589	0.0662	0.9599	0.0540	0.9543
1.592	0.0450	0.9232	0.0404	0.9232	0.0313	0.9199
1.688	0.0257	0.8839	0.0207	0.8839	0.0146	0.8823
1.758	0.0119	0.8360	0.0066	0.8360	0.0030	0.8357
1.801	0.0032	0.7642	0.0000	0.8149	0.0000	0.8473
1.817	0.0000	0.0000	0.0000	0.8382	0.0000	0.8677
2.326		0.9394		0.9466		0.9544
3.383		0.9779		0.9800		0.9828
4.631		0.9391		0.9907		0.9920
5.952		0.9936		0.9951		0.9958
7.308		0.9958		0.9972		0.9976
8.682		0.9970		0.9983		0.9985
10.067		0.9978		0.9989		0.9991

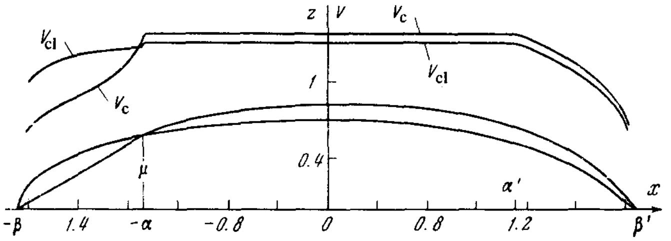

Fig. 5

shape of the free surface obtained for the flow past a wedge and past an elliptic arc without allowance for a gravitational force.

Two different flows past an elliptic arc were calculated with allowance for gravitational force. The results are compiled in Table 2. The case G = 0.7848 for S = 1 m corresponds to an oncoming-flow velocity of 5 m/sec. The quantity H denotes the maximum cavern height.

It should be noted that if the straight line z = R, and not the streamline, is taken as C_2 (Fig. 1), for an arbitrary R we have a cavitating channel flow. For comparison, Fig. 7 shows the shapes of the free surface which develop in the cavitating flow past a wedge, according to the mirror scheme, in a half-plane (solid line) and in a channel (dashed line) with a width of 2R = 4. The cavitation number σ is 0.9309 and 1.9704, respectively.

4. The numerical method developed for solving two-dimensional problems can be applied to the solution of analogous axisymmetric problems. Let us examine, for example, Ryabushinskii's problem for the axisymmetric case, with allowance for surface tension.

In this case, instead of (1.1), (1.2), we have the following equations:

$$\frac{\partial\psi}{\partial x} = -r\frac{\partial\varphi}{\partial r} = -rV_r, \qquad \frac{\partial\psi}{\partial r} = r\frac{\partial\varphi}{\partial x} = rV_x, \tag{4.1}$$

$$P = 1 - V^2. \tag{4.2}$$

If the cavern is assumed to be a shell of uniform strength, the conditions on it may be written as:

$$V^2 = \sigma + 1 + T(\varkappa_1 + \varkappa_2), \tag{4.3}$$

where T is the specific tension, and $\varkappa_1$ and $\varkappa_2$ are the principal curvatures of the cavern surface.

Conditions (1.4) now take the form

$$\psi = {}^1/_2 r^2 \text{ on } C_1,\ C_3, \quad \psi = {}^1/_2 R^2 \text{ on } C_2, \quad \psi = 0 \text{ on } C \tag{4.4}$$

while in the equation for the function $z(x, \psi) = r^2(x, \psi)$ (see (1.5)), only the coefficient c undergoes a change,

$$c = 4z + z_x^2. \tag{4.5}$$

The flow rate is then expressed by the formula

$$V^2 = \frac{4z + z_x^2}{z z_\psi^2}. \tag{4.6}$$

The problem reduces to the solution of Eq. (1.5) within the range $\omega'(-\gamma \le x \le \gamma,\ 0 \le \psi \le R^2/2)$ and the solution of (1.5) with allowance for (4.5), for the boundary conditions (Fig. 2)

$$z = 2\psi \text{ on } \Gamma,\ z = F_q(x) = W_q^2(x) \quad \text{on } \Gamma_q, \quad q = 1, 2. \tag{4.7}$$

Here, $W_q(x)$ stands for the cavitator and closing-curve equations and conditions (4.3) on Γ'.

Eqs. (1.5), (4.5) are integrated using one of the fractional step schemes (1.9), (1.10), the initial integral surface being given by the formula

$$z^\circ(x, \psi) = 2\left(1 - \frac{z^\circ(x, 0)}{R^2}\right)\psi + z^\circ(x, 0). \tag{4.8}$$

Here, $z^\circ(x, 0) = r^{\circ 2}(x, 0)$, $r^\circ(x, 0)$ is the equation for the zero streamline. The initial shape of the free boundary is taken in the form of the surface of revolution of a circular arc with the center on the r-axis, which has a smooth contact with the cavitator and closing curve.

The free boundary is corrected by once-through filtering, along the boundary, of relation (4.3) written in the following form:

$$4z^{k+1} + z_x^k z_x^{k+1} - \left[\sigma + 1 + 4T\frac{z_x^2 + z(2 - z_{xx})}{(4z + z_x^2)^{3/2}}\right]^k \times$$
$$\times (z z_\psi)^k (\xi z^{k+1} + z_\psi^k - \xi z^k) = 0. \tag{4.9}$$

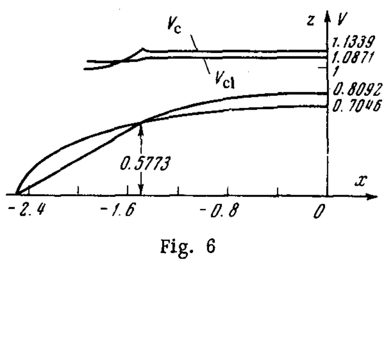

Fig. 6

Table 2

G	H	$V(0, H)$	σ
0.0000	0.5740	1.2368	0.5297
0.1962	0.5705	1.2328	0.6318
0.7848	0.5612	1.2225	0.9349

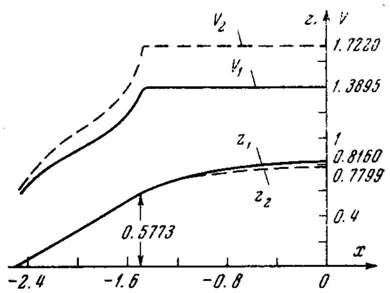

Fig. 7

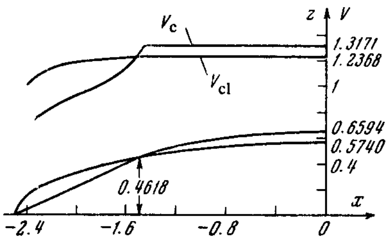

Fig. 8

Here, z_x and z_ψ have the form (1.14), while σ^k is evaluated from the formula (Fig. 8)

$$\sigma^k + 1 = \left[\frac{1}{z z_\psi^2}\left(4z + F^{12}(-\alpha)\right) - 4T\,\frac{z_x^2 + z\,(2 - z_{xx})}{(4z + z_x^2)^{3/2}}\right]^k_{x=-\alpha+h}. \qquad (4.10)$$

Figure 8 gives the free surface and the velocity curve for cavitators in the form of a right cone with a cone angle of $\pi/3$ and of an ellipsoid or revolution; $\alpha = 1.5$, $\beta = 2.5$ is taken in each case. Calculations were performed without allowance for surface tension ($T = 0$). It was found that the influence of surface-tension forces that develop at the water-air interface on the flow characteristics is negligible at oncoming-flow velocities above 10 m/sec.

Figure 9 gives a plot of the cavitation number σ vs. the cavern half-length L for a right circular cone with a cone angle of 2 arctg(0.3) for $\beta - \alpha = 1$.

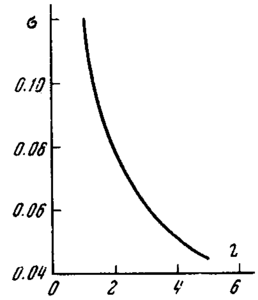

Fig. 9

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